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Sampled-data-based consensus of continuous-time systems with limited data rate

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Abstract: Leader-following consensus is investigated for second-order continuous-time multi-agent systems with limited communication data rate in this study. The network communication topology is directed and static. Using the sampled position and velocity data, a corresponding discrete-time system, whose leader-following consensus is proved to be equivalent to that of the original system, is given. A distributed protocol is proposed based on dynamic encoding and decoding. A necessary and sufficient condition is given for the existence of such communication and control protocol to ensure the achievement of leader-following consensus, if the upper bound of the communication data rate is predetermined. Two algorithms are also presented to achieve the leader-following consensus if the communication data rate is unfixed or limited, respectively. Finally, simulation examples are given to verify and illustrate the theoretical analysis.

1 Introduction

The distributed cooperative control of multi-agent systems has been an intense area of research in recent years, motivated by many applications in control engineering, physics and biology. In cooperative control, consensus problem in networked multi-agent systems has been a topic of significant interest. If there exists a single leader or multiple leaders, the leader-following consensus or containment of multi-agent system will be considered [1–4]. In [1], the distributed consensus tracking algorithms without velocity measurements are proposed under both fixed and switching network topologies. In [3], necessary and sufficient criteria which guarantee the achievement of containment of first-order system are given for both static and dynamic leaders.

In engineering applications, because of the unreliability of information channels, the sensing ability of each agent or the cost, only sampled-data at discrete sampling instants is available for control synthesis in many cases, though the systems are continuous process. Generally, multi-agent system with sampled-data control is often considered in second-order system [5–9]. In [5], the consensus of double-integrator dynamics with only sampled positions is considered. In [6], necessary and sufficient conditions for multi-agent system via sampled control are provided. In [7], not only the sampled current position data, but also the post position data is utilised to design the protocol. Chen *et al*. [9] addresses the consensus with directed topology and static quantiser. In [10], sampled-data-based average-consensus of continuous-time first-order integrator agent is considered, in which the measurements of the neighbour states are corrupted by random measurement noises. The consensus based on sampled-data is also studied in observer-based system [11], event-triggered transmission strategy [12], asynchronous hybrid event-time driven interaction [13], multi-tracking [14], asymptotic bounded consensus tracking [15, 16].

On the other hand, in practice, if the information of agents is real number, it is obvious that the communication channels need to be unlimited. While in the digital communication channel, the data should be integer, often binary signal, and the communication channel has a channel bandwidth. Li *et al*. [17] studies the average consensus control of discrete-time multi-agent system with undirected graph, and shows that average consensus is achievable with a finite number of quantisation levels. Extensions of [17] are further discussed in time-varying topology [18], high-order systems [19], and quantised-observer case [20]. All the above literatures are considered in undirected topology. Li and Xie [21] extends the results of [17] for undirected network to general case of directed network.

There is few literature to study the case of continuous-time multi-agent system with limit communication data rate, which is more practical. In continuous-time system, the communication data rate is determined by the amount of transmitted data per second, which means that not only the channel bandwidth, but also the sampled period needs to be considered. Mu and Liu [22] firstly considers the containment control with limited communication data rate in continuous-time multi-agent systems, and necessary and sufficient criteria which ensure the property of consensus are given. Because of the structure constraint of the protocol in [22], there exists a lower bound of the communication data rate to ensure the achievement of containment. In this paper, we further study the leader-following consensus of second-order continuous-time linear multi-agent system with limited communication data rate. The sampled position and velocity states are quantised into digital signals by the given encoders and decoders, and sent to their neighbours. A necessary and sufficient condition of leader-following consensus with sampled-data and limited data rate is given. Moreover, two algorithms are also given to design the proper sampled period and encoder-decoder for unfixed or any given limited communication data rate, respectively.

Our main challenge is to design the sampled period, encoder-decoder pairs and sampled-data based protocol for directed communication topology to achieve the leader-following consensus. It needs to be emphasised that this paper is not a simple combination of the aforementioned literatures on sampled-data and limited data rate. There are two fundamental difficulties. One is that the sampled-data based control systems are hybrid and with both continuous-time and discrete-time signal, so not only the states at sampling instants, but also the final effect of the control law on the original continuous system should be considered, which is what the designers are really concerned about. The other is that the communication channels between the agents have limited capacity, so only finite bytes of data can be sent per second, which may lead to unbounded quantisation errors. To overcome these difficulties, we design a corresponding discrete system, whose leader-following consensus is proved to be equivalent to that of the sampled-data based continuous system with limited data rate, and using it as a basis, two algorithms are given to design the sampled-data-based disturbed leader-following consensus protocol with an encoder-decoder. Compared with [6], limited data rate is considered in this paper. In [20], the topology is undirected and the system is discrete-time, while in this paper, the topology is directed and the system is continuous-time. Compared with [22], the system is second-order and the leader is dynamic in this paper. Moreover, the consensus can be achieved under any given limited communication data rate, while [22] needs the data rate to exceed a bound. In [9], the consensus errors converge to a neighbourhood of the origin because of the constraints of static quantiser, while in this paper, the consensus errors accurately converge to the origin because of the dynamic encoder-decoder.

This paper is organised as follows. In Section 2, the model of network and the basic material on graph theory needed in this paper are introduced. In Section 3, the protocol with sampled states and encoder-decoder are proposed, and the equivalence of consensus between the continuous-time system and sampled system is given. In Section 4, the case of unfixed communication data rate is considered. In Section 5, for limited communication data rate, a necessary and sufficient condition is given to achieve the leader-following consensus, and an algorithm is proposed to design the proper protocol with sampled period and encoder-decoder. Finally, simulations are provided to illustrate the effectiveness of the theoretical results in Section 6.

Notation: Given a matrix A, $\Lambda(A)$ and $r(A)$ denote its eigenvalue set and spectral radius, respectively. 1_n denotes the n-dimension column vector with all ones, I_n denotes the $n \times n$ identity matrix. $\| \cdot \|$ and $\| \cdot \|_\infty$, respectively, represent the standard L_2 and L_∞ norms on vectors or their induced norms on matrices. It is easy to get that $\forall x \in \mathbb{R}^n$, $\| x \|_\infty \le \| x \| \le \sqrt{n} \| x \|_\infty$. $\mathrm{diag}\{A_1, \ldots, A_n\}$ represents the diagonal. Given a complex number λ, $\mathrm{Re}(\lambda)$, $\mathrm{Im}(\lambda)$ and $|\lambda|$ are the real part, imaginary part and modulus of λ, respectively. For a given positive real number y, $\lceil y \rceil$ denotes the smallest integer bigger than or equal to it, $\log_2(y)$ denotes the logarithm of y with base 2.

2 Model description and problem formulation

To investigate the consensus problem of multi-agent systems, algebraic graph theory is a useful theory. We first introduce some basic definitions in the algebraic graph theory [23]. A weighted directed graph is denoted by $\mathscr{G} = \{\mathscr{V}, \mathscr{E}, \mathscr{A}\}$, with the set of nodes $\mathscr{V} = \{1, 2, \ldots, n\}$, the set of edges $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$, and the weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$. The edges of the graph are denoted by $e_{ij} = (j, i)$, where j is called the parent vertex or neighbour of i. The adjacency elements associated with the edges are positive, i.e. an edge $(j, i) \in \mathscr{E}$ if and only if $a_{ij} > 0$, which means node i can receive the data from node j. Self-edge (i,i) is not allowed, i.e. $(i, i) \notin \mathscr{E}$. A directed path between node i_1 and node i_k is meant a sequence of distinct edges with the form $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k)$, and a weak path, either (i_s, i_{s+1}) or $(i_{s+1}, i_s) \in \mathscr{E}$. A graph is strongly connected if there exists a directed path between any two different nodes of the graph, and is weakly connected if any two vertices can be jointed by a weak path. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A directed spanning tree is a directed tree, which consists of all the nodes and some edges in $\mathscr{G}$.

The in-degree of node i is denoted as $\deg_{in}(i) = \sum_{j=1}^{n} a_{ij}$. Denote the Laplacian Matrix of the graph $\mathscr{G}$ by $\mathscr{L} = \mathscr{D} - \mathscr{A}$, where $\mathscr{D} = \mathrm{diag}\{\deg_{in}(1), \deg_{in}(2), \cdots, \deg_{in}(n)\}$. Furthermore, 0 is the simple eigenvalue of $\mathscr{L}$ if and only if $\mathscr{G}$ is connected for undirected or has a spanning tree for digraph.

To study a leader-following problem, we also concern another graph $\bar{\mathscr{G}}$ associated with the system consisting of n followers and one leader (labelled 0). Similarly, a diagonal matrix $\mathscr{B} \in \mathbb{R}^{n \times n}$ is defined to be a leader adjacency matrix associated with $\bar{\mathscr{G}}$, whose diagonal elements $b_i = a_{i0}$ for some constant $a_{i0} > 0$ if the leader is a neighbour of node i and $b_i = 0$, otherwise. Denote $\mathscr{H} = \mathscr{L} + \mathscr{B}$. Let $d_{\max} = \max_i \{\mathscr{H}_{ii}\}$, where $\mathscr{H}_{ii}$ are the diagonal elements of $\mathscr{H}$. For $\bar{\mathscr{G}}$, if there is a path in $\bar{\mathscr{G}}$ from node 0 to every node i, we say that the leader is globally reachable. The set of node i's neighbours is denoted by $\mathscr{N}_i = \{j | (j, i) \in \mathscr{E}, j \ne i\}$.

Lemma 1 [24]: All of the eigenvalues of the matrix $\mathscr{H} = \mathscr{L} + \mathscr{B}$ have positive real parts if and only if the leader 0 is globally reachable in graph $\bar{\mathscr{G}}$.

In this paper, we consider a group of second-order continuous-time multi-agent systems with double-integrator dynamics,

$$\begin{cases} \dot{x}_i(t) = v_i(t) \\ \dot{v}_i(t) = u_i(t) \end{cases} i = 0, 1, \ldots, n. \tag{1}$$

For convenience we assume one-dimensional movement, i.e. $x_i(t)$, $v_i(t) \in \mathbb{R}$ represent the position and velocity of agent i, respectively, and they are transmitted as the agent's output. $u_i(t) \in \mathbb{R}$ represents the control input, that is, consensus protocol. Specially, consensus protocol of the leader is zero, that is, $u_0(t) \equiv 0$ and $v_0(t) \equiv v_0(0)$.

Definition 1 Leader-following consensus:: The multi-agent system (1) is said to achieve leader-following consensus if its solution satisfies $\lim_{t \to \infty} \| x_i(t) - x_0(t) \| = 0$, $\lim_{t \to \infty} \| v_i(t) - v_0(t) \| = 0$, $i \in \mathscr{V}$, for any initial condition.

In this paper, we aim to design an encoding-decoding scheme and an interaction protocol using a finite number of bits per second of data rate to achieve the leader-following consensus.

3 Protocol design

For the second-order multi-agent system, some researchers considered the following consensus protocol at every moment [25, 26]:

$$\begin{aligned} u_i(t) = a &\sum_{j \in \mathscr{N}_i} a_{ij}(x_j(t) - x_i(t)) \\ &+ b \sum_{j \in \mathscr{N}_i} a_{ij}(v_j(t) - v_i(t)), \quad i \in \mathscr{V} \end{aligned} \tag{2}$$

where $a > 0$ and $b > 0$ are the coupling gains.

In above case, all the position and velocity data of the agents needs to be obtained in time, i.e. for every $t > 0$, the agent i needs to get the exact state information from its neighbours. However, In engineering applications, exact information often cannot be obtained, while the quantised data, usually symbolic data, is communicated between the neighbours at each discrete time instant.

Let $h > 0$ be the sampling period and $k = 0, 1, \ldots$ be the sampling instants. The protocol is given by a piecewise constant function which identically equals a constant, denoted by $u(kh)$, $t \in [kh, (k+1)h)$. Then, the sampled system of (1) can be written as

$$\begin{cases} x_i(t) = x_i(kh) + (t - kh)v_i(kh) + \dfrac{(t - kh)^2}{2} u_i(kh) \\ v_i(t) = v_i(kh) + (t - kh)u_i(kh) \end{cases} \tag{3}$$

$t \in [kh, (k+1)h)$. From (3), we can see that the position $x_i(t)$ and the velocity $v_i(t)$ in the continuous-time system (1) are not piecewise constant functions. The corresponding discrete-time system is given as follows,

$$\begin{cases} x_i[k+1] = x_i[k] + hv_i[k] + \dfrac{h^2}{2} u_i[k] \\ v_i[k+1] = v_i[k] + hu_i[k] \end{cases} \tag{4}$$

in which $x_i(kh)$, $v_i(kh)$ and $u_i(kh)$ are, respectively, denoted by $x_i[k]$, $v_i[k]$ and $u_i[k]$, $\forall k \in \mathbb{Z}_{\ge 0}$, for the sake of simplicity.

In digital communication network, only symbolic data can be communicated between agents through digital channels. Thus for each digital channel, the state of the sender is encoded into symbolic data and then transmitted. After the symbolic data is received, the receiver uses a decoder to get an estimate of the sender's state. Similar to [17, 18], the encode, quantiser and decoder are given as follows, respectively. The encoder Φ_j associated with jth agent is defined as follows. For all $k = 1, 2, \ldots,$

$$\begin{cases} \hat{x}_j[0] = \hat{v}_j[0] = 0, \\ \hat{x}_j[k] = \hat{x}_j[k-1] + h\hat{v}_j[k-1] + g(k-1)\Delta_{x_j}[k], \\ \hat{v}_j[k] = \hat{v}_j[k-1] + g(k-1)\Delta_{v_j}[k], \\ \Delta_{x_j}[k] = q\Big(\dfrac{1}{g(k-1)}(x_j[k] - \hat{x}_j[k-1] - h\hat{v}_j[k-1])\Big), \\ \Delta_{v_j}[k] = q\Big(\dfrac{1}{g(k-1)}(v_j[k] - \hat{v}_j[k-1])\Big), \end{cases} \tag{5}$$

where $x_j[k]$, $v_j[k]$ are the inputs of encoder Φ_j, and $\hat{x}_j[k]$, $\hat{v}_j[k]$ are the internal states of encoder Φ_j. $\Delta_{x_j}[k]$ and $\Delta_{v_j}[k]$ are the outputs of the encoder Φ_j, which will be broadcasted through the digital channels. $g(k) > 0$ is a scaling function. Here we choose the scaling function $g(k) = g_0\gamma^k$, with $g_0 > 0$ and $\gamma \in (0, 1)$ being design parameters. $q(\cdot)$ is a finite-level quantiser as follows

$$q(x) = \begin{cases} 0, & -1/2 < x < 1/2, \\ i, & \dfrac{2i-1}{2} \le x < \dfrac{2i+1}{2}, i = 1, 2 \ldots L, \\ L, & x \ge \dfrac{2L+1}{2}, \\ -q(-x), & x \le -1/2, \end{cases} \tag{6}$$

where the quantisation level is given as $2L + 1$ with a positive integer $L > 0$. According to the construction of the quantiser, we can get that $|x - q(x)| \le 1/2$, if $|x| < L + 1/2$.

Remark 1: If the leader-following consensus is achieved, the error $x_j[k] - \hat{x}_j[k]$ tends to zero, as $k \to \infty$. Therefore, the scaling function $g(t)$ should meet the following two properties. Firstly, it should decay gradually, such that the data of the agent can be estimated continuously. Secondly, it should be large enough to make sure that the quantiser will not be saturated.

When the agent i receives the symbolic data $\Delta_{x_j}[k]$ and $\Delta_{x_j}[k]$ sent by agent j through the communication channel $(j, i) \in \mathscr{E}$, the position and velocity data of the agent j can be estimated by the following decoder Ψ_{ji} associated with the directed channel (j,i). For all $k = 1, 2, \ldots,$

$$\begin{cases} \hat{x}_{ji}[0] = \hat{v}_{ji}[0] = 0, \\ \hat{x}_{ji}[k] = \hat{x}_{ji}[k-1] + h\hat{v}_{ji}[k-1] + g(k-1)\Delta_{x_j}[k], \\ \hat{v}_{ji}[k] = \hat{v}_{ji}[k-1] + g(k-1)\Delta_{v_j}[k], \end{cases} \tag{7}$$

in which $\hat{x}_{ji}[k]$ and $\hat{v}_{ji}[k]$ are the outputs of the decoder Ψ_{ji}.

Definition 2: The communication data rate is defined as the average number of bits per second passing in the communication channel, i.e. $R_\mathrm{d} = \lceil \log_2(2L+1) \rceil / h$.

Remark 2: The consensus of the discrete-time multi-agent system with limited data rate is considered in [17, 21], in which we only need to consider the quantisation level to get the communication data rate because of the fixed sampled period. However, when the communication data rate is considered in continuous-time case, in order to calculate the data rate, not only the quantisation level $2L + 1$, but also the sampled period h needs to be considered.

In this paper, we aim at designing a distributed protocol to achieve leader-following consensus based on sampled-data and quantised communications. For sampled-data based continuous-time system (1), using the encoder, quantiser, and decoder given above, the distributed protocol is proposed as follows,

$$\begin{aligned} u_i(t) = u_i[k] = a &\sum_{j \in \mathscr{N}_i} a_{ij}(\hat{x}_{ji}[k] - \hat{x}_i[k]) \\ &+ b \sum_{j \in \mathscr{N}_i} a_{ij}(\hat{v}_{ji}[k] - \hat{v}_i[k]), \end{aligned} \tag{8}$$

$i \in \mathscr{V}, t \in [kh, (k+1)h)$, in which we can see that the protocol only depends on the states of its own encoder and the decoders associated with the directed channels from its neighbours.

Remark 3: According to (5) and (7), we can get that

$$\hat{x}_{ji}[k] = \hat{x}_j[k], \quad \hat{v}_{ji}[k] = \hat{v}_j[k],$$

for all $k = 0, 1, \ldots, j \in \mathscr{N}_i, i \in \mathscr{V}$. Thus, the above protocol (8) can be written as

$$\begin{aligned} u_i[k] = a &\sum_{j \in \mathscr{N}_i} a_{ij}(\hat{x}_j[k] - \hat{x}_i[k]) \\ &+ b \sum_{j \in \mathscr{N}_i} a_{ij}(\hat{v}_j[k] - \hat{v}_i[k]), \quad i \in \mathscr{V}. \end{aligned} \tag{9}$$

Now, we consider the relation between the leader-following consensus of the system (1) and (4). In many engineering practice, the leader, the reference state or the control target of the system may have the time-varying velocity, such as [15, 16, 27, 28], thus the following lemma considers the system with the time-varying velocity without loss of generality.

Lemma 2: System (1) achieves leader-following consensus if and only if system (4) achieves leader-following consensus.

Proof: Necessity can be easily get because discrete-time states $\{x_i[k]\}_{k \ge 0}$ and $\{v_i[k]\}_{k \ge 0}$ are the sub-sequences of $\{x_i(t)\}_{t \ge 0}$ and $\{v_i(t)\}_{t \ge 0,}$ respectively.
The sufficiency will be proved by two steps.
First, we show that $\lim_{k \to \infty} |u_i[k] - u_0[k]| = 0$, $\forall i \in \mathscr{V}$, if the system (4) achieves leader-following consensus.
According to the definition of leader-following consensus, we can get that $\lim_{k \to \infty} x_i[k] = x_0[k]$, $\lim_{k \to \infty} v_i[k] = v_0[k]$, $i \in \mathscr{V}$. Together with the fact

$$x_i[k+1] = x_i[k] + hv_i[k] + \frac{1}{2}h^2 u_i[k],$$

$$x_0[k+1] = x_0[k] + hv_0[k] + \frac{1}{2}h^2 u_0[k],$$

we have

$$\begin{aligned} x_i[k+1] - x_0[k+1] &= (x_i[k] - x_0[k]) \\ &\quad + h(v_i[k] - v_0[k]) + \frac{1}{2}h^2(u_i[k] - u_0[k]). \end{aligned}$$

Thus, $\lim_{k \to \infty} |u_i[k] - u_0[k]| = 0$, $i \in \mathscr{V}$.
Secondly, we show that the sufficiency holds. For any given $t \in [kh, (k+1)h)$, according to the term (3), it yields that

$$\begin{aligned} |x_i(t) - x_0(t)| \le{} & |x_i[k] - x_0[k]| + |v_i[k] - v_0[k]|(t - kh) \\ & + \frac{1}{2}|u_i[k] - u_0[k]|(t - kh)^2, \end{aligned} \tag{10}$$

and

$$\begin{aligned} & |v_i(t) - v_0(t)| \\ & \le |v_i[k] - v_0[k]| + |u_i[k] - u_0[k]|(t - kh). \end{aligned} \tag{11}$$

Together with the fact $\lim_{k \to \infty} |u_i[k] - u_0[k]| = 0$, we have

$$\lim_{k \to \infty} |x_i(t) - x_0(t)| = 0, \quad \lim_{k \to \infty} |v_i(t) - v_0(t)| = 0.$$

The proof is completed. □

Remark 4: It seems to be a trivial conclusion that the consensus for sampled system (4) is equivalent to that for continuous system (1). However, in the previous literatures, such as [3, 5, 7], the consensus for the continuous system was demonstrated directly or the equivalence between the consensus of continuous system and sampled system was not provided. According to Lemma 2, we can study the leader-following consensus of system (4) to replace that of system (1).

4 Convergence analysis with unfixed communication data rate

This section is devoted to achieve the leader-following consensus when the data rate of the communication channel is not preset, which means that the quantiser parameter L and sampling period h can be chosen freely. While the convergence analysis of the proposed distributed control law with limited data rate will be studied in the next section. To this end, we introduce the following notations

$$X[k] = (x_1[k], \ldots, x_n[k])^{\mathrm{T}}, \quad V[k] = (v_1[k], \ldots, v_n[k])^{\mathrm{T}},$$
$$U[k] = (u_1[k], \ldots, u_n[k])^{\mathrm{T}}, \quad \hat{X}[k] = (\hat{x}_1[k], \ldots, \hat{x}_n[k])^{\mathrm{T}},$$
$$\hat{V}[k] = (\hat{v}_1[k], \ldots, \hat{v}_n[k])^{\mathrm{T}}, \quad \delta_X[k] = X[k] - x_0[k]1_n,$$
$$\delta_V[k] = V[k] - v_0[k]1_n, \quad E_X[k] = X[k] - \hat{X}[k],$$
$$E_V[k] = V[k] - \hat{V}[k], \quad E_{x_0}[k] = (x_0[k] - \hat{x}_0[k])1_n,$$
$$E_{v_0}[k] = (v_0[k] - \hat{v}_0[k])1_n,$$

$$G(h, a, b) = \begin{bmatrix} I_n - \frac{1}{2}h^2 a\mathscr{H} & hI_n - \frac{1}{2}h^2 b\mathscr{H} \\ -ha\mathscr{H} & I_n - hb\mathscr{H} \end{bmatrix}. \tag{12}$$

The following assumptions are given.

Assumption 1: There exist some constants $C_X, C_V, C_\delta > 0$, s.t. $|x_i[0]| \le C_X$, $|v_i[0]| \le C_V$, for $i = 0, 1, \ldots, n$, and $\| \delta_X(0) \| \le C_\delta$, $\| \delta_V(0) \| \le C_\delta$.

Assumption 2: In the topology $\bar{\mathscr{G}}$, the leader is globally reachable.

Some useful lemmas are given as follows.

Lemma 3 [29]: For any given $A \in \mathbb{R}^{n \times n}$, $\epsilon > 0$, we have

$$\| A^k \| \le M\eta^k, \quad \forall k \ge 0 \tag{13}$$

where $M = \sqrt{n}(1 + (2/\epsilon))^{n-1}$, $\eta = r(A) + \epsilon \| A \|$.

Lemma 4 [30]: Given a complex coefficient polynomial of order two as follows

$$g(s) = s^2 + (\xi_1 + i\gamma_1)s + \xi_0 + i\gamma_0,$$

where ξ_1, γ_1, ξ_0 and γ_0 are real constants. Then, $g(s)$ is stable if and only if $\xi_1 > 0$ and $\xi_1\gamma_1\gamma_0 + \xi_1^2\xi_0 - \gamma_0^2 > 0$.

The following lemma is proposed to investigate the spectral radius of the matrix $G(h, a, b)$.

Lemma 5: $r(G(h, a, b)) < 1$ if and only if the following two conditions hold:

(i) $h < 2b/a$,

(ii) $\forall \mu_i \in \Lambda(\mathscr{H})$, $8(\mathrm{Im}(\mu_i))^2 \le (2\mathrm{Re}(\mu_i) - bh|\mu_i|^2)|\mu_i|^2(1/a)(2b - ah)^2$.

Proof: From (12), we have

$$\begin{bmatrix} I_n & 0 \\ -\frac{ha\mathscr{H}}{\lambda - 1} & I_n \end{bmatrix}\begin{bmatrix} I_n & -\frac{1}{2}hI_n \\ 0 & I_n \end{bmatrix}(\lambda I_{2n} - G)$$
$$= \begin{bmatrix} (\lambda - 1)I_n & -\frac{1}{2}(\lambda + 1)hI_n \\ 0 & (\lambda - 1)I_n + hb\mathscr{H} + \frac{\lambda + 1}{2(\lambda - 1)}h^2 a\mathscr{H} \end{bmatrix}.$$

Then we can get that

$$\det(\lambda I_{2n} - G) = \left|(\lambda - 1)^2 I_n + \left(\left(\frac{1}{2}ah + b\right)\lambda + \left(\frac{1}{2}ah - b\right)\right)h\mathscr{H}\right|$$
$$= \prod_{\mu \in \Lambda(\mathscr{H})} \left((\lambda - 1)^2 + \left(\left(\frac{1}{2}ah + b\right)\lambda + \left(\frac{1}{2}ah - b\right)\right)h\mu\right)^{\varsigma(\mu)},$$

in which $\varsigma(\mu)$ is the algebraic multiplicity of μ, for every $\mu \in \Lambda(\mathscr{H})$.

Let $s = (\lambda + 1)/(\lambda - 1)$, then we can get that

$$\det(\lambda I_{2n} - G) = \prod_{\mu \in \Lambda(\mathscr{H})} \left(\frac{ah^2\mu}{(s - 1)^2}\left(s^2 - \left(1 - \frac{2b}{ah}\right)s + \frac{2}{ah}\left(\frac{2}{h\mu} - b\right)\right)\right)^{\varsigma(\mu)}.$$

Thus, $r(G(h, a, b)) < 1$ if and only if for any $\mu \in \Lambda(\mathscr{H})$, all of the eigenvalues of the following function are in the left half complex plane

$$\begin{aligned} f_\mu(s) &= s^2 - \left(1 - \frac{2b}{ah}\right)s + \frac{2}{ah}\left(\frac{2}{h\mu} - b\right) \\ &= s^2 - \left(1 - \frac{2b}{ah}\right)s + \left(\frac{4}{ah^2} \cdot \frac{\mathrm{Re}(\mu)}{|\mu|^2} - \frac{2b}{ah}\right) \\ &\quad + \frac{4}{ah^2} \cdot \frac{\mathrm{Im}(\mu)}{|\mu|^2} i. \end{aligned} \tag{14}$$

According to Lemma 4, one can get that this lemma holds. □

In the Following we will give the first main result of this paper.

Theorem 1: For the continuous-time multi-agent system (1), suppose that Assumptions 1 and 2 hold. The constants $h > 0$, a, b are chosen to ensure $r(G(h, a, b)) < 1$. Let $\rho \in [r(G(h, a, b)), 1)$ and $\gamma \in (\rho, 1)$. For any given L, such that

$$L \ge D_*(h, a, b)\left(\sqrt{2n}\frac{D_*(h, a, b)}{\gamma(\gamma - \rho)} + \frac{1}{\gamma}\right) + \frac{1}{2\gamma}(h + 1) - \frac{1}{2}, \tag{15}$$

where $D_*(h, a, b) = d_{\max}h(a + b)\max\{h, 2\}$, let

$$g_0 \ge \max\left\{\frac{C_X + hC_V}{L + 1/2}, \frac{C_V}{L + 1/2}, \frac{C_\delta\gamma(\gamma - \rho)}{D_*(h, a, b)}\right\}. \tag{16}$$

Then under the protocol (9) with the sampled period h, the encoder (5), decoder (7) and $(2L + 1)$-level quantiser (6), the system (1) achieves leader-following consensus.

Proof: According to the Lemma 2, we only need to prove that the discrete-time system (4) achieves leader-following consensus under the protocol (9). The key idea of the following is to show that the quantiser (6) is always unsaturated with every $k \ge 0$.

According to the terms (4), (5), (7) and (9), the dynamics of the asymptotic errors are proposed as follows

$$\delta_X[k+1] = \delta_X[k] + h\delta_V[k] + \frac{1}{2}h^2 U[k]$$
$$= \left(I - \frac{1}{2}h^2 a\mathscr{H}\right)\delta_X[k] + \left(hI - \frac{1}{2}h^2 b\mathscr{H}\right)\delta_V[k] + \frac{1}{2}h^2 E_{\mathrm{ALL}}(a,b,k),$$
$$\delta_V[k+1] = \delta_V[k] + hU[k]$$
$$= -ha\mathscr{H}\delta_X[k] + (I - hb\mathscr{H})\delta_V[k] + hE_{\mathrm{ALL}}(a,b,k),$$

where $E_{\mathrm{ALL}}(a,b,k) = a\mathscr{H}E_X[k] + b\mathscr{H}E_V[k] - a\mathscr{H}E_{x_0}[k] - b\mathscr{H}E_{v_0}[k]$.

Moreover, the dynamics of the estimate errors are given as

$$E_X[k+1] = X[k+1] - \hat{X}[k] - h\hat{V}[k] - g(k)Q\left(\frac{1}{g(k)}(X[k+1] - \hat{X}[k] - h\hat{V}[k])\right), \tag{17}$$

$$E_V[k+1] = V[k+1] - \hat{V}[k] - g(k)Q\left(\frac{1}{g(k)}(V[k+1] - \hat{V}[k])\right), \tag{18}$$

in which the product quantiser $Q(\cdot)$ is provided by $Q(\cdot) = (q(\cdot), q(\cdot), \cdots, q(\cdot))^{\mathrm{T}}$, whose dimension is compatible with its input.

Denote $\bar{\delta}_X[k] = (1/g(k))\delta_X[k]$, and similarly give the definitions of $\bar{\delta}_V[k]$, $\bar{E}_X[k]$, $\bar{E}_V[k]$ and $\bar{E}_{\mathrm{ALL}}[k]$. Then we have

$$\gamma\bar{\delta}_X[k+1] = \left(I - \frac{1}{2}h^2 a\mathscr{H}\right)\bar{\delta}_X[k] + \left(hI - \frac{1}{2}h^2 b\mathscr{H}\right)\bar{\delta}_V[k] + \frac{1}{2}h^2\bar{E}_{\mathrm{ALL}}(a,b,k), \tag{19}$$

$$\gamma\bar{\delta}_V[k+1] = -ha\mathscr{H}\bar{\delta}_X[k] + (I - hb\mathscr{H})\bar{\delta}_V[k] + h\bar{E}_{\mathrm{ALL}}(a,b,k), \tag{20}$$

$$\gamma\bar{E}_X[k+1] = R_X[k] - Q(R_X[k]), \tag{21}$$

$$\gamma\bar{E}_V[k+1] = R_V[k] - Q(R_V[k]), \tag{22}$$

where $R_X[k] = (1/g(k))(X[k+1] - \hat{X}[k] - h\hat{V}[k])$, and $R_V[k] = (1/g(k))(V[k+1] - \hat{V}[k])$.

It is obvious that $\| \bar{E}_X[k+1] \|_\infty \le 1/2\gamma$, $\| \bar{E}_{x_0}[k+1] \|_\infty \le 1/2\gamma$, $\| \bar{E}_V[k+1] \|_\infty \le 1/2\gamma$, and $|\bar{E}_{v_0}[k+1] \|_\infty \le 1/2\gamma$, when $\| R_X[k] \|_\infty \le L + 1/2$, $\| R_V[k] \|_\infty \le L + 1/2$. Furthermore, we have

$$\| \bar{E}_{\mathrm{ALL}}(a,b,k) \|_\infty \le \frac{2}{\gamma}(a+b)d_{\max}. \tag{23}$$

Now we prove that the quantiser will never be saturated, which means that we will prove $\| R_X[k] \|_\infty < L + 1/2$ and $\| R_v[k] \|_\infty < L + 1/2, \forall k \in \mathbb{Z}_{\ge 0}$, by mathematical induction.

Firstly, because Assumption 1 holds, then

$$\| R_X[0] \|_\infty = \frac{1}{g_0}\| X[1] - \bar{X}[0] - h\bar{V}[0] \|_\infty \le \frac{1}{g_0}(C_X + hC_V) \le L + \frac{1}{2},$$
$$\| R_V[0] \|_\infty \le \frac{1}{g_0}C_V \le L + \frac{1}{2}.$$

Thus, when $k = 0$, the quantiser is unsaturated.

Second, for any given non-negative integer κ, suppose that when $k \le \kappa$, the quantiser is unsaturated, which means that $\sup_{k \le \kappa} \| R_X[k] \|_\infty \le L + 1/2$, $\sup_{k \le \kappa} \| R_V[k] \|_\infty \le L + 1/2$. Below, we prove that the quantiser is unsaturated when $k = \kappa + 1$.

$$R_X[k] = \bar{E}_X[k] + h\bar{E}_V[k] + \frac{1}{2}h^2(-a\mathscr{H}\bar{\delta}_X[k] - b\mathscr{H}\bar{\delta}_V[k] + \bar{E}_{\mathrm{ALL}}[k])$$
$$= -\frac{1}{2}h^2 a\mathscr{H}\bar{\delta}_X[k] - \frac{1}{2}h^2 b\mathscr{H}\bar{\delta}_V[k] + \left(\frac{1}{2}h^2\bar{E}_{\mathrm{ALL}} + \bar{E}_X[k] + h\bar{E}_V[k]\right),$$
$$R_V[k] = -ha\mathscr{H}\bar{\delta}_X[k] - hb\mathscr{H}\delta_V[k] + \left(h\bar{E}_{\mathrm{ALL}} + \bar{E}_V[k]\right).$$

It follows that

$$\| R[k] \|_\infty \le D_*(h,a,b)\left(\left\| \begin{bmatrix}\delta_X[k]\\ \delta_V[k]\end{bmatrix} \right\|_\infty + \frac{1}{\gamma}\right) + \frac{1}{2\gamma}(h+1), \tag{24}$$

where $R[k] = \left[R_X^{\mathrm{T}}[k], R_V^{\mathrm{T}}[k]\right]^{\mathrm{T}}$. The last ' $\le$' holds because of the term (23).

By the terms (19) and (20), we can get that

$$\gamma\begin{bmatrix}\bar{\delta}_X[\kappa+1]\\ \bar{\delta}_V[\kappa+1]\end{bmatrix} = G(h,a,b)\begin{bmatrix}\bar{\delta}_X[\kappa]\\ \bar{\delta}_V[\kappa]\end{bmatrix} + \frac{h}{2}\begin{bmatrix}h\\ 2\end{bmatrix} \otimes I_n\bar{E}_{\mathrm{ALL}}(a,b,\kappa).$$

According to Lemma 3, There exist constants $\epsilon \in (0, (1 - r(G(h,a,b)))/\| G(h,a,b) \|)$, $\eta = r(G(h,a,b)) + \epsilon \| G(h,a,b) \| \in (0,1)$ and $M = \sqrt{n}(1 + (2/\epsilon))^{n-1}$, such that

$$\| G(h,a,b)^k \| < M\eta^k.$$

Denote $\bar{\delta}[k] = [\bar{\delta}_X^{\mathrm{T}}[k], \bar{\delta}_V^{\mathrm{T}}[k]]^{\mathrm{T}}$, then

$$\gamma \| \bar{\delta}[\kappa+1] \| \le \| G(h,a,b)\bar{\delta}[\kappa] \| + \frac{1}{2}\sqrt{2n}h \max\{2,h\}\frac{2}{\gamma}(a+b)d_{\max}$$
$$\le r(G(h,a,b)) \| \bar{\delta}[\kappa] \| + D_*(h,a,b)\frac{\sqrt{2n}}{\gamma}.$$

According to Lemma 5, choose γ and ρ such that $r(G(h,a,b)) < \rho < \gamma < 1$, which follows that

$$\begin{aligned} &\| \bar{\delta}[\kappa+1] \| \\ &\le \frac{\rho}{\gamma}\| \bar{\delta}[\kappa] \| + D_*(h,a,b)\frac{\sqrt{2n}}{\gamma^2} \\ &\le \left(\frac{\rho}{\gamma}\right)^{\kappa+1}\| \bar{\delta}[0] \| + \frac{\sqrt{2n}}{\gamma^2}D_*(h,a,b)\sum_{i=0}^{\kappa}\left(\frac{\rho}{\gamma}\right)^i \\ &\le \frac{C_\delta}{g_0}\left(\frac{\rho}{\gamma}\right)^{\kappa+1} + \frac{\sqrt{2n}}{\gamma(\gamma-\rho)}D_*(h,a,b)\left(1 - \left(\frac{\rho}{\gamma}\right)^{\kappa+1}\right) \\ &\le \sqrt{2n}\max\left\{\frac{C_\delta}{g_0}, \frac{D_*(h,a,b)}{\gamma(\gamma-\rho)}\right\}. \end{aligned} \tag{25}$$

Together with terms (15), (16) and (24), we can get $\| R[\kappa+1] \|_\infty \le L + 1/2$.

Thus, the quantiser is confirmed to be unsaturated at time $\kappa + 1$. By induction, the $(2L+1)$-level quantiser will never be saturated.

Noting the term (25), we have

$$\sup_{k \ge 0} \| \bar{\delta}_X[k] \| < \infty, \quad \sup_{k \ge 0} \| \bar{\delta}_V[k] \| < \infty.$$

Together with the definitions of $\bar{\delta}_X[k]$ and $\bar{\delta}_V[k]$, it follows that the system (4) achieves leader-following consensus. By Lemma 2 and

Remark 4, we can get that the continuous-time system (1) achieves the leader-following consensus.□

Remark 5: Here, the scaling function $g(t)$ is designed off-line. According to the term (16), the choice of g_0 depends on the upper bound C_X, C_V, C_δ, the communication graph and the protocol gains. This is a conservative selection, and in practice, smaller g_0 may be available.

Remark 6: If the quantisation level $2L+1$ could be chosen freely, Theorem 1 says that the system can achieve leader-following consensus by using the scaling function $g(t)$ and the communication data rate which is given as $R_\text{d} = \lceil \log_2(2L+1) \rceil / h$. It is worth noting that the communication data rate R_d given here and the quantisation level $2L+1$ with the term (15) are conservative estimates, and in practice, smaller R_d and $2L+1$ may be available. It would be an interesting future topic and open problem to find the fundamental lower bound for the quantisation level. In fact, we will propose an algorithm to achieve leader-following consensus for any given communication data rate in the next section.

Remark 7: The existences of the control gain a, b and the sampled period h are based on the inequalities of Lemma 5. According to the above theorem, we can get that smaller γ leads to a better asymptotic convergence factor. For the extreme case, as γ approaches $r(G(h,a,b))$ from the right hand side, according to the term (15), the quantisation level $2L+1$ goes to infinity, which means that the communication data rate also goes to infinity, when the sampled period h remains unchanged.

For the unfixed communication data rate, which means that we can choose the quantisation level $2L+1$ and sampling period h freely, we can get a simple algorithm to choose proper parameters to make sure that the system (1) achieves the leader-following consensus.

Algorithm 1:

Step 1: Choose h, a, b, such that $r(G(h,a,b)) < 1$.
Step 2: Choose $\rho \in [r(G(h,a,b)), 1)$, $\gamma \in (\rho, 1)$.
Step 3: Choose a positive integer L, such that the term (15) holds.
Step 4: Choose $g_0 > 0$, such that the term (16) holds.

Remark 8: The existence of the parameters *h,a,b* which ensures $r(G(h,a,b)) < 1$ will be given in following Lemma 6.

5 Leader-following consensus with limited data rate

In this section, leader-following consensus with limited communication data rate is considered, which means that R_d is pre-determined by the capacity of transmission channels between agents and their neighbours.

The following lemma is proposed to analyse the property of the matrix $G(h,a,b)$.

Lemma 6: Suppose that Assumption 2 holds, then there exist constants $h>0$, $a>0$ and $b>0$ such that the inequalities of Lemma 5 hold. Moreover, if $\bar{h}$, $\bar{a}$ and $\bar{b}$ ensure that the inequalities of Lemma 5 hold, then for any given $\alpha > 0$, $h = \alpha\bar{h}$, $a = \bar{a}/\alpha^2$, $b = \bar{b}/\alpha$ also ensure that the inequalities of Lemma 5 hold. Moreover, we have

$$\Lambda(G(h,a,b)) = \Lambda(G(\bar{h},\bar{a},\bar{b})). \quad (26)$$

Proof: According to Lemma 1, when Assumption 2 holds, we can get that all the eigenvalues of matrix $\mathscr{H}$ have positive real parts. So the solvability of the inequities of Lemma 5 can be given similarly as [6].

If $\bar{h}$, $\bar{a}$ and $\bar{b}$ ensure that the inequalities of Lemma 5 holds, it is easy to verify that the *h,a,b* also satisfy the inequalities.
Moreover, we have

$$G\left(\alpha\bar{h}, \frac{\bar{a}}{\alpha^2}, \frac{\bar{b}}{\alpha}\right) = \begin{bmatrix} \alpha I_n & 0 \\ 0 & I_n \end{bmatrix} G(\bar{h},\bar{a},\bar{b}) \begin{bmatrix} \frac{1}{\alpha} I_n & 0 \\ 0 & I_n \end{bmatrix}$$

Then $\Lambda(G(h,a,b)) = \Lambda(G(\bar{h},\bar{a},\bar{b}))$. □

The main result of this section can be summarised as follows.

Theorem 2: Suppose Assumption 1 holds, there exists protocol (8) with limited communication data rate for the continuous-time multi-agent system (1) to achieve leader-following consensus, if and only if the leader of $\bar{\mathscr{G}}$ is globally reachable.

Proof Sufficiency:: According to Lemma 5, Lemma 6 and Theorem 1, we can get that there exist proper constants $\bar{h}$, $\bar{a}$, $\bar{b}$, and L, such that $r(G(\bar{h},\bar{a},\bar{b})) < \rho < \gamma < 1$ and the terms (15) and (16) hold.
For any given $\alpha > 0$, denote $h = \alpha\bar{h}$ as the sampled period, $a = \bar{a}/\alpha^2$ and $b = \bar{b}/\alpha$ as the protocol gains. According to Lemma 6, we can get $r(G(h,a,b)) = r(G(\bar{h},\bar{a},\bar{b}))$, which means that the corresponding parameters ρ and γ do not change with α.
Suppose $\bar{R}$ is a pre-determined upper bound of the communication data rate R_d because of the capacity of transmission channel. In order to ensure that the quantiser is unsaturated, the data rate R_d needs to satisfy

$$\begin{aligned} \bar{R} \geq R_\text{d} &= \frac{\lceil \log_2(2L+1) \rceil}{h} \\ &\geq \frac{1}{\alpha\bar{h}} \log_2\left(D_*(h,a,b)\left(\sqrt{2n}\max\left\{C_\delta, \frac{D_*(h,a,b)}{\gamma(\gamma-\rho)}\right\} + \frac{1}{\gamma}\right)\right. \\ &\quad \left. + \frac{1}{2\gamma}(h+1)\right) \end{aligned} \quad (27)$$

Obviously, as α goes to infinity, the right hand side of (27) tends to 0. Then for any given $\bar{R}$, there exist proper parameters such that the term (27) holds, which means that the leader-following consensus can be achieved.
Necessity: If the leader of the graph $\bar{\mathscr{G}}$ is not globally reachable, there might be at least one follower such that it could not receive any information sent by leader. It follows that this follower is independent of the position of the leader. This means that leader-following consensus of the multi-agent system may not be achieved.
The proof is completed. □

Now, for the pre-determined upper bound $\bar{R}$ of the communication data rate, we can get a simple algorithm to choose proper sampled period, encoder-decoder, and protocol gain to make sure that the system (1) achieves the leader-following consensus.

Algorithm 2:

Step 1: Choose $\bar{h}$, $\bar{a}$, $\bar{b}$, such that $r(G(\bar{h},\bar{a},\bar{b})) < 1$.
Step 2: Choose $\rho \in [r(G(\bar{h},\bar{a},\bar{b})), 1)$, $\gamma \in (\rho, 1)$.
Step 3: Choose $\alpha > 0$ large enough, such that (27) holds.
Step 4: Let the sampled period $h = \alpha\bar{h}$, the protocol gains $a = \bar{a}/\alpha^2$ and $b = \bar{b}/\alpha$.
Step 5: Choose a positive integer L, such that the terms(15) holds.
Step 6: Choose $g_0 > 0$, such that the term (16) holds.

6 Example

In this section, some examples are given to demonstrate the validity of the proposed theoretical results.

Example 1: Consider the multi-agent system (1) with a leader-follower network with four followers. The communication graph is

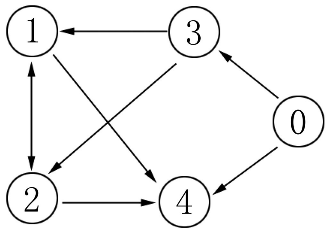

Fig. 1 *Communication graph, which has a leader labelled 0 and 4 followers labelled* 1, 2, 3, 4

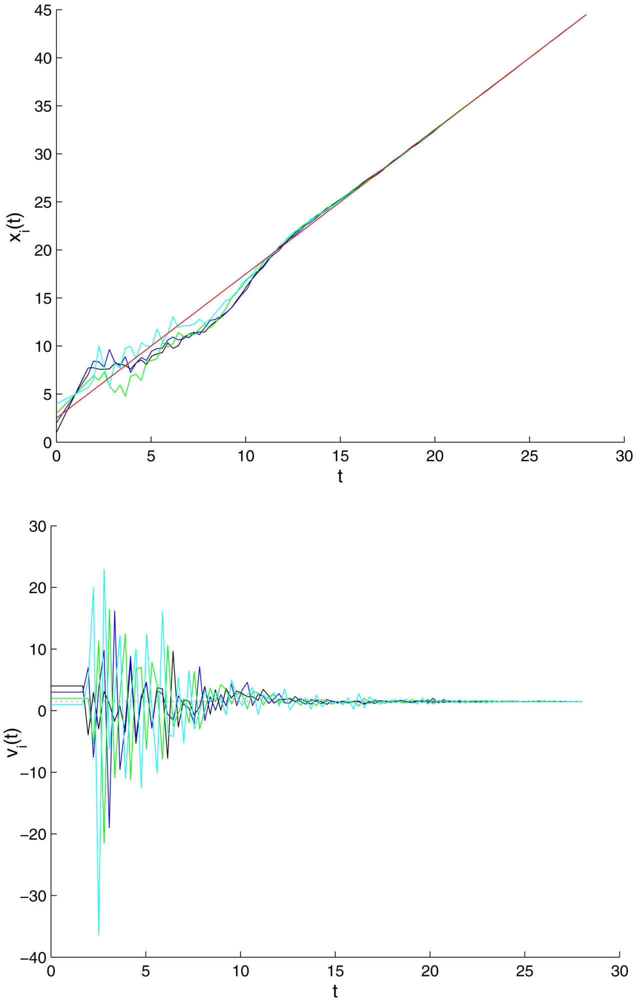

Fig. 2 *Trajectories of states $x_i(t)$ and $v_i(t)$ with unfixed communication data rate*

given in Fig. 1 with $0-1$ weight, which means that $a_{ij}=1$, if $(j,i)\in\mathscr{E}$, otherwise, $a_{ij}=0$.

If the communication data rate could be chosen freely, according to Theorem 1 and Algorithm 1, we can choose $h=0.28$, $a=1$, $b=2$, then $r(G(h,a,b))=0.8864<1$. Choose $\rho=0.8865,\gamma=0.9332$. It follows that quantisation level $L=1159$ and the communication data rate $R_{\rm d}=39.9260$. The state trajectories of the leader and all the followers with unfixed communication data rate are given in Fig. 2, in which we can see that the leader-following consensus is achieved.

Example 2: Still consider the communication graph of Fig. 1 with 0–1 weights. Now we consider the situation that the communication data rate is limited.

Firstly, similar as Example 1, we choose $\bar h=0.28$, $\bar a=1,\bar b=2$, then $r(G(\bar h,\bar a,\bar b))=0.8864<1$. Choose $\rho=0.8865,\gamma=0.9332$.

By Theorem 2 and Algorithm 2, if the upper bound $\bar R$ of the communication data rate is given as $\bar R=10$, we can choose $\alpha=3.75$, which follows that sampled period $h=1.05$, protocol

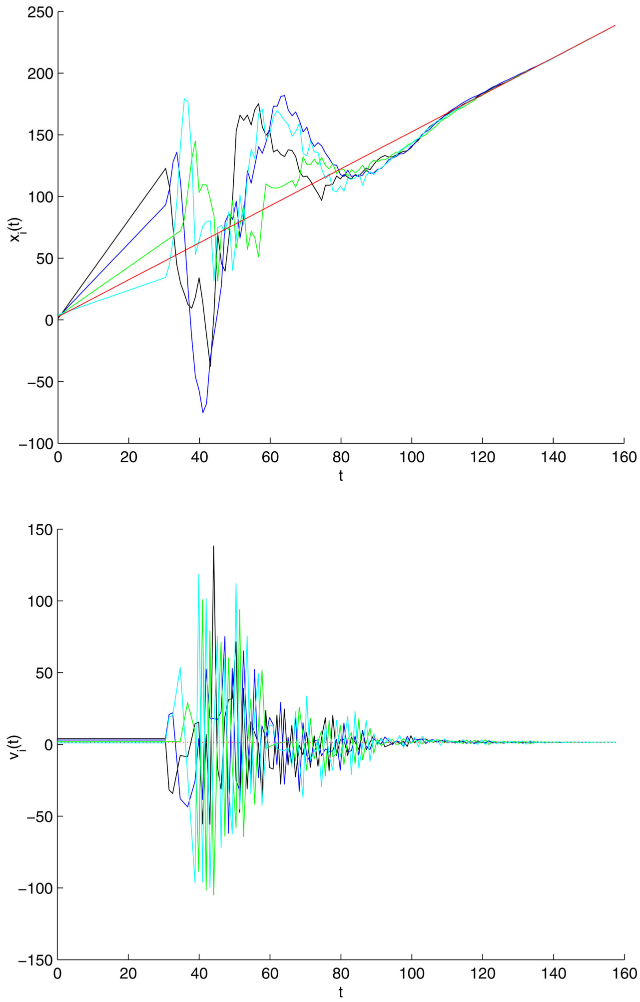

Fig. 3 *Trajectories of states $x_i(t)$ and $v_i(t)$ with $\bar R=10$*

gains $a=0.7111\times10^{-2}$, $b=0.5333$ and quantisation level $L=663$. The communication data rate $R_{\rm d}$ is given as $R_{\rm d}=9.8799<\bar R$. The simulation result is shown in Fig. 3. We can see that all followers converge to the leader and the leader-following consensus is achieved.

If the upper bound $\bar R$ of the communication data rate is given as $\bar R=1$, by Algorithm 1, sampled period, quantisation level and the communication data rate are given as $h=16.24$, $L=34409$, and $R_{\rm d}=0.9896<\bar R$. Fig. 4 shows the simulation results.

Comparing the above two cases and the corresponding simulation results, the smaller communication data rate we choose, the more slowly that the followers tend to the leader.

7 Conclusion

Motivated by the situation that the agents can only receive the quantised data and the communication channels have limited data rate, we have studied the joint effects of the sampled states and dynamic encoder-decoder on the leader-following consensus of continuous-time second-order multi-agent system, in which both of the position and velocity data needs to be sampled and quantised. Using a lemma, the leader-following consensus of continuous-time system is proved to be equipment to its corresponding discrete-time system. A necessary and sufficient condition and two algorithms to choose the proper sampled period and encoder-decoder have been given. Finally, the theoretical results have been verified by simulations.

8 Acknowledgments

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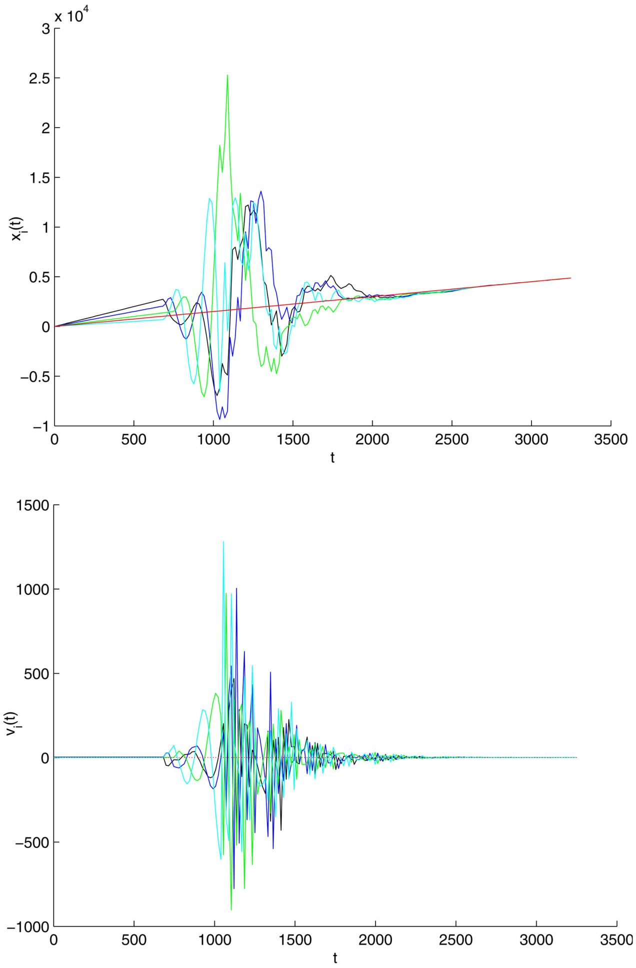

Fig. 4 *Trajectories of states $x_i(t)$ and $v_i(t)$ with $\bar{R} = 1$*

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